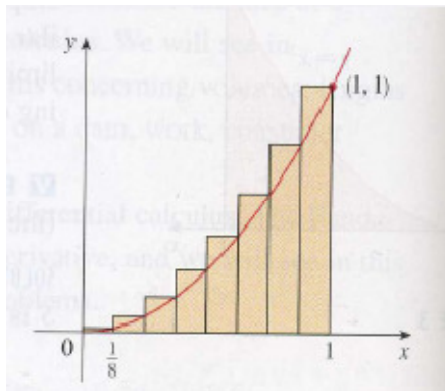
Math 141 –Section 5.1, 5.2, 5.3 –Week 1

1.1 Area under a curve

Area under a curve can be expressed as the area of many rectangles added together. As seen below.



The more rectangles there are, the better the approximation.

This can be done for any curve that is continuous on the interval [a, b] and above the x-axis.

A common procedure for doing this:

1) Divide the curve into *n* rectangles of equal width, each rectangle has a width of $\frac{b-a}{n}$ and is denoted by Δx. ($\Delta x = \frac{b-a}{n}$) Each width is also known as a subinterval
2) In each subinterval, choose a point (x-coordinate) within the subinterval. This point is denoted by x^*_i and is called a **sample point**. (The subscript *i* means that you can have x^*_1, x^*_2, x^*_3.. where *i* = 1,2 3, 4... n) This sample point could be any number within the subinterval but usually the left endpoint, right endpoint, or the midpoint of each subinterval is chosen to simplify the calculations.

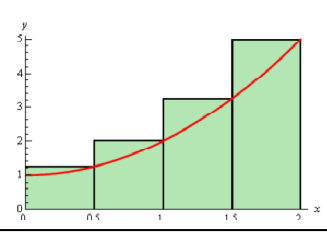
   Right endpoint: Denoted by x_i. x_i = a + $i \bullet \Delta x$, where *i = 1, 2, 3, 4... n*

   Left endpoint: Denoted by x_{i-1}. x_{i-1} = a + $i \bullet \Delta x$, where *i = 0, 1, 2, 3, 4... n*

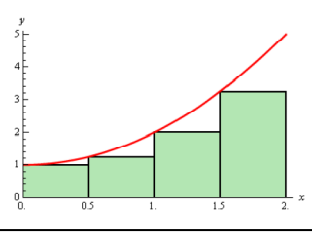
   Midpoint: Denoted by $\overline{x}_i$. $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

   This point is sub into the equation that represents the curve and becomes the height of the rectangle in that specific subinterval: $f(x^*_i)$

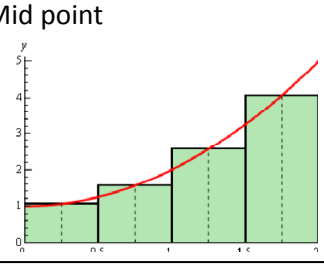
Right endpoint



Left endpoint



Mid point



3) Area of each rectangle is given by $f(x^*_i)\bullet\Delta x$

4) We add up all those rectangles by using the summation/sigma notation.

End

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

Begin

Denote total area of all the rectangles by S_n

$$S_n = \sum_{i=1}^{n} f(x^*_i)\bullet\Delta x$$

This expression is a special case of what's called the **Riemann Sums**

5) Next, we indicate how big *n* is by taking $n \to \infty$ via limit.

$$S_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x^*_i)\bullet\Delta x\right)$$

This can be also expressed equally as

$$S_n = \lim_{\Delta x\to 0}\left(\sum_{i=1}^{n} f(x^*_i)\bullet\Delta x\right)$$

<u>1.2 General case of Riemann sums</u>

Dividing area into equal subintervals is not necessary. It is simply done to make calculations easier and is therefore a special case of Riemann sums.

In the general case of Riemann sums, subintervals do not have to be equal. As long the largest subinterval, called the **norm,** and denoted by $\|p\|$, goes to zero via limit , Riemann sums holds. Thus, the general expression of Riemann sums is as follows.

$$\sum_{i=1}^{n} f(x^*_i)\bullet\Delta x_i$$

(subscript is added to Δx to show that it is not equal width)

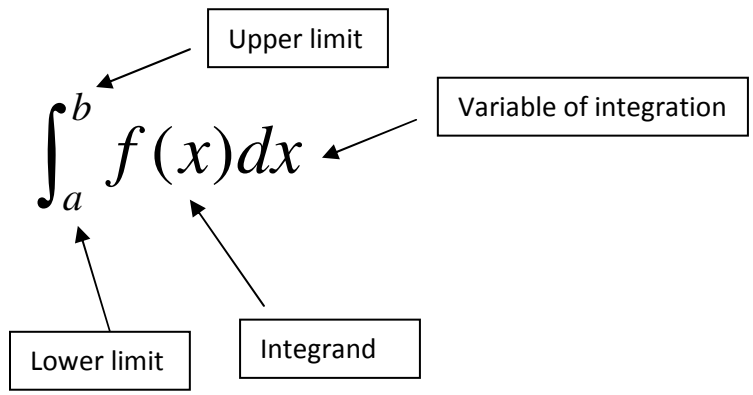
Riemann sums with limit added in:

$$\lim_{\|p\|\to 0}\sum_{i=1}^{n} f(x^*_i)\bullet\Delta x_i$$

<u>1.3 Definite Integral</u>

Riemann sums with limit is called a definite integral and is defined as the following

$$\int_a^b f(x)dx = \lim_{\|p\|\to 0}\sum_{i=1}^{n} f(x^*_i)\bullet\Delta x_i$$



The following parts have equivalent meaning:

$$\int_a^b = \lim_{\|p\|\to 0}\sum_{i=1}^{n}$$

f(x) = $f(x^*_i)$

dx = Δx_i

In essence, Integral is the limit of Riemann sums.

<u>1.4 Rules and Formulas of summation:</u>

1. Constant term rule: $\sum_{i=1}^{n} c = c + c + c.. + c = nc$

2. Sum rule: $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

3. Scalar multiple rule: $\sum_{i=1}^{n} ca_k = c\sum_{i=1}^{n} a_k$

4. Linearity rule: $\sum_{i=1}^{n} (ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$

5. Subtotal rule: If 1 < m < n then

$$\sum_{i=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

<u>1.5 Properties of Integrals</u>

1. Definite integral at a point: $\int_a^a f(x)dx = 0$
2. Opposite of a definite integral: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. Sum/difference rule: $\int_a^b [f(x) \pm g(x)] = \int_a^b f(x) \pm \int_a^b g(x)$
4. Constant rule: $\int_a^b cdx = c(b-a)$
5. Scalar multiple rule: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
6. Linearity rule: $\int_a^b [rf(x) + sg(x)]dx = r\int_a^b f(x)dx + s\int_a^b g(x)dx$
7. Dominance rule: $f(x) \geq g(x)$ then, $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
8. Subdivision rule: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

1.6 Dummy variable

As long the upper and lower limits of integration and the integrand stay the same, the variable of integration can be anything and won't affect the answer. In other words:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz$$

x, t, z are known as dummy variables since they play no role in the result. They serve only as different variables to avoid confusion.

1.7 Area and definite integrals

Area can be expressed directly as a definite integral only if $f(x) \geq 0$. If *f(x)* have parts above and below the x-axis, then the net area can be calculated by dividing into parts above the x-axis and below the x-axis. Do the definite integral for each and add the resultant area. (Area = area above x-axis – area below x-axis)

Sometimes, area can be used to aid in evaluating the integral if the integral represents the area of some common geometric figure. For example:

$$\int_a^b \sqrt{9-x^2}\,dx$$

Let *f(x)* = $\sqrt{9-x^2}$ *. Then y =* $\sqrt{9-x^2}$ *.* y^2 *= 9-* x^2 *or* $x^2 + y^2 = 9$*, where is the formula of a circle with radius of 3.*

Therefore, f(x) is simply a semicircle centered at the origin with radius of 3. Area of a circle is πr^2 *so the area of semicircle is simply* $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 3^2 = \frac{9\pi}{2}$.

Thus, $\int_a^b \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$.

1.8 Integrability:

Generally, continuous functions on [a, b] are integrable on [a,b].

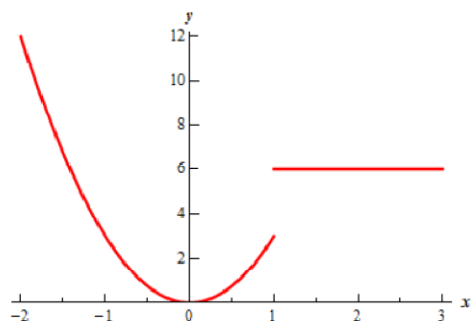
However, sometimes, if function *f* has finitely many points of discontinuity on the interval [a, b] and the value f(x) doesn't go positive or negative infinity then *f* can be integrable. An example of this kind of function are piecewise functions. To find the value, we break up the interval at the point where discontinuity occurs and then integrate each section.

For example:

$$f(x)=\begin{cases} 6 \text{ if } x>1 \\ 3x^2 \text{ if } x\le 1 \end{cases}$$

Find $\int_{-2}^{3} f(x)dx$

Here is the graph of the function:



We did the interval [-2, 3] into two intervals at x = 1 (where is it discontinuous) and use the subdivision rule of integrals $\int_a^c f(x)dx=\int_a^b f(x)dx+\int_b^c f(x)dx$.

Thus, $\int_{-2}^{3} f(x)dx=\int_{-2}^{1} f(x)dx+\int_{1}^{3} f(x)dx$

<u>1.9 The fundamental theorm of calculus</u>

1. if $g(x)=\int_a^x f(t)dt$ then $g'(x)=f(x)$ or $\frac{d}{dx}\int_a^x f(t)dt=f(x)$
2. $\int_a^b f(x)dx=F(b)-F(a)$, where $F'=f$

Part 1 relates the integral calculus and differential calculus by stating that if *g* give the area of *f,* then the derivative of *g* give the function *f.* This makes sense if one relates it in terms of velocity and displacment. Suppose f(x) give an object's velocity. Then g(x) is the area between the velocity curve and the x-axis. In other words, g(x) give an object's displacement. Thus, if we differntiate g(x), the result will be f(x), the velocity function.

Part 2 says that the definite integral $\int_a^b f(x)dx$ can be evaluated by finding the antiderivative F(x) and evaluate the F(x) at the limits of the integration a and b. F(b) – F(a) can be also written as $F(x)\Big]_a^b$.

1.10 Techniques for using the fundemental theorm of calculus

1. When the variable x is in the lower limit rather than the upper limit

Tactic: switch them around and put – sign in front

Example:

$$\frac{d}{dx}\left[\int_x^1 e^{2t}\,dt\right]$$

$$\frac{d}{dx}\left[\int_x^1 e^{2t}\,dt\right]$$
$$=-\frac{d}{dx}\left[\int_1^x e^{2t}\,dt\right]$$
$$=-e^{2x}$$

2. When the upper limit is not x

Tactic: use implict differentiation.

Example:

$$\frac{d}{dx}\left[\int_{x^2}^1 \frac{t^4+1}{t^2+1}dt\right]$$

$$\frac{d}{dx}\left[\int_{x^2}^1 \frac{t^4+1}{t^2+1}dt\right]$$
$$=-\frac{d}{dx}\left[\int_1^{x^2} \frac{t^4+1}{t^2+1}dt\right]$$
$$=-\frac{d}{du}\left[-\int_1^{u} \frac{t^4+1}{t^2+1}dt\right]\bullet\frac{du}{dx}$$
$$=-\left(\frac{u^4+1}{u^2+1}\right)\bullet\frac{du}{dx}$$
$$=-\left(\frac{(x^2)^4+1}{(x^2)^2+1}\right)\bullet 2x$$
$$=-2x\left(\frac{x^8+1}{x^4+1}\right)$$

3. Both upper and limits are functions of x

Tactic: Break up the integral using this property: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Example:

$$\frac{d}{dx}\left[\int_{\sqrt{x}}^{3x} t^2 \sin(1+t^2)dt\right]$$

$$\frac{d}{dx}\left[\int_{\sqrt{x}}^{3x} t^2 \sin(1+t^2)dt\right]$$

$$= \frac{d}{dx}\left[\int_{\sqrt{x}}^{0} t^2 \sin(1+t^2)dt + \int_0^{3x} t^2 \sin(1+t^2)dt\right]$$

$$= \frac{d}{dx}\left[-\int_0^{\sqrt{x}} t^2 \sin(1+t^2)dt + \int_0^{3x} t^2 \sin(1+t^2)dt\right]$$

$$= \frac{d}{du}\left[-\int_0^{u} t^2 \sin(1+t^2)dt\right] \bullet \frac{du}{dx} + \frac{d}{dz}\left[\int_0^{z} t^2 \sin(1+t^2)dt\right] \bullet \frac{dz}{dx}$$

$$= -\left(u^2 \sin(1+u^2)\right)\left(\frac{1}{2\sqrt{x}}\right) + \left(z^2 \sin(1+z^2)\right)(3)$$

$$= \frac{-x\sin(1+x)}{2\sqrt{x}} + \left((3x)^2 \sin(1+(3x)^2)\right)(3)$$

$$= \frac{-x\sin(1+x)}{2\sqrt{x}} + \left(9x^2 \sin(1+9x^2)\right)(3)$$

$$= \frac{-x\sin(1+x)}{2\sqrt{x}} + 27x^2 \sin(1+9x^2)$$

$$= \frac{-\sqrt{x}\sin(1+x)}{2} + 27x^2 \sin(1+9x^2)$$